

# The Klein Bottle, a Continuous Dictionary for Distributions of High-Contrast Image Patches

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## Topological Inference

Point cloud

$$X \subseteq \mathbb{R}^n$$



Persistent homology



Betti numbers

$$\beta_1, \dots, \beta_k$$

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Topological information  $\implies$  Ingenuity  $\implies$  Paramt.  $f \in Emb(T, \mathbb{R}^n)$ ,  $f(T) \sim X$   
 Models:  $f \in Map(X, T)$

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- After mean centering, contrast normalization and a linear change of coordinates, can regard  $X \subseteq S^7$ .
- 50% of the points in  $X$  have the topology of a Klein bottle, modeled by the space

$$\mathcal{K} = \{p(x, y) = c(ax + by) + d(ax + by)^2, a^2 + b^2 = c^2 + d^2 = 1\}.$$

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**Idea:** If  $h_k(I) : \mathcal{K} \rightarrow \mathbb{R}$  is the underlying PDF, Fourier Analysis on  $L^2(\mathcal{K})$  yields a compact representation  $v_k(I)$  of  $h_k(I)$ .

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**Application:** Texture discrimination and classification via  $\{v_k(I)\}_k$ .

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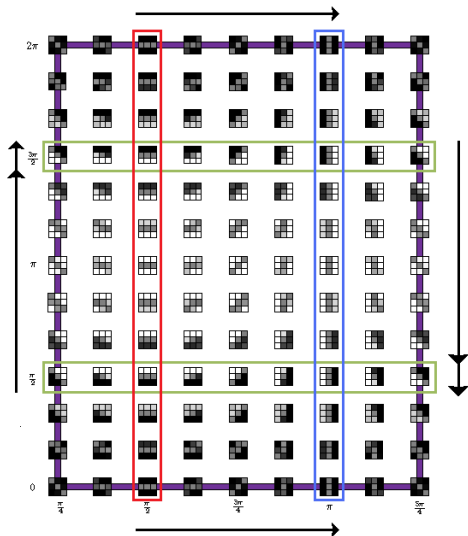
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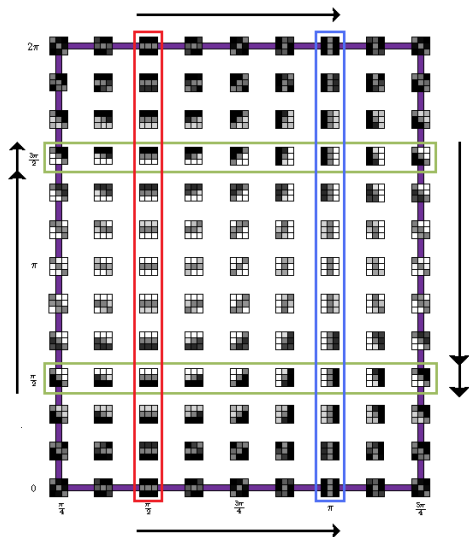
All results were obtained using the MATLAB R2009b software.

# Projection onto $\mathcal{K}$

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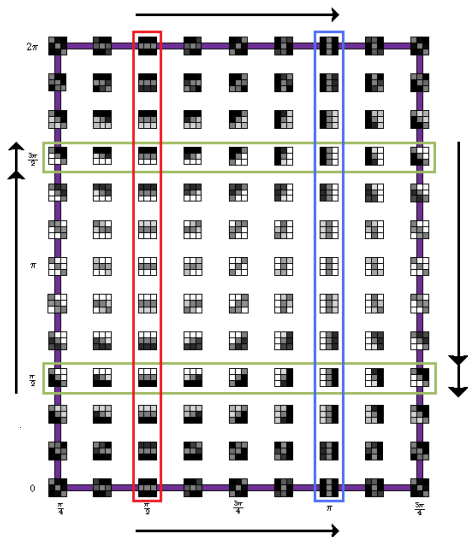


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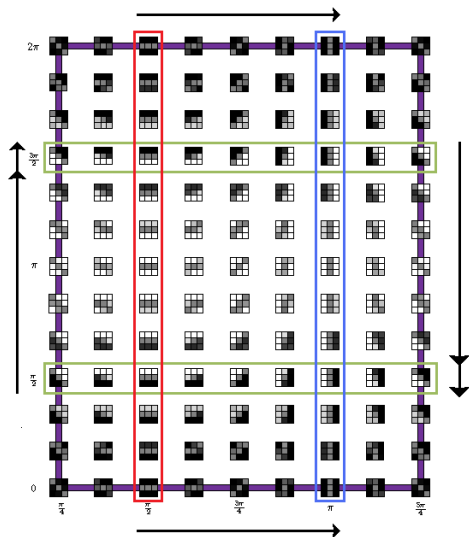
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- Vertical component: linear versus quadratic contribution.

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- Horizontal coordinate: predominant direction.
- Vertical component: linear versus quadratic contribution.
- Given  $f : [-1, 1]^2 \rightarrow \mathbb{R}$ , its **direction**  $\theta \in [0, \pi)$  should be so that  $f$  is, in average, as constant as possible along the  $(-\sin \theta, \cos \theta)$  direction.

## Theorem

Let  $f : [-1, 1]^2 \rightarrow \mathbb{R}$  be differentiable and let  $Q_f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be

$$Q_f(v) = \iint_{[-1,1]^2} \langle \nabla f, v \rangle^2 dx dy.$$

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- ② Let  $j : S^1 \rightarrow \mathbb{R}P^1$  be the quotient map and assume the eigenvalues of  $A_f$ , the matrix representing  $Q_f$ , are distinct. Then the direction map

$$\text{Dir}(f) = j\left(\arg \max_{\|v\|=1} Q_f(v)\right)$$

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is well defined and continuous in the  $C^1$ -topology. Moreover, the maximum is attained at the eigenvectors of  $A_f$  with largest eigenvalue.

Projection  $P \mapsto c(ax + by) + d(ax + by)^2$ , discrete case...

Given a  $k \times k$  patch  $P : [-1, 1]^2 \longrightarrow \mathbb{R}$ , approximate  $\nabla P$  and  $\iint \cdot dx dy$ .

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so one can write the degree 2 polynomial approximation

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where  $r_1 = \frac{\langle P, ax+by \rangle}{\|ax+by\|^2}$  and  $r_2 = \frac{\langle P, (ax+by)^2 \rangle}{\|(ax+by)^2\|^2}$ . Let  $r = (r_1, r_2)$ .

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- 4 If  $\|r\| \geq s_k$  (threshold) let  $(c, d) = r/\|r\|$ .

## An orthonormal basis for $L^2(K)$

- Let  $\mu$  be the  $\mathbb{Z}/2\mathbb{Z}$ -action on  $S^1 \times S^1$  given by  $1 \cdot (z, w) = (-z, \bar{w})$ .

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- The map

$$\begin{aligned} \varphi : S^1 \times S^1 &\longrightarrow \mathcal{K} \\ (\alpha, \beta) &\longmapsto p_{\alpha, \beta} \end{aligned}$$

$p_{\alpha, \beta}(x, y) = c(ax + by) + d(ax + by)^2$ , (here  $a + bi = e^{i\alpha}$ ) is  $\mu$ -invariant ( $\varphi \circ \mu(g, z, w) = \varphi(z, w)$ ) and induces a homeomorphism  $\varphi_* : \mathcal{K} \longrightarrow \mathcal{K}$  from the orbit space.

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- Fourier Analysis on  $L^2(S^1 \times S^1) \Rightarrow$  orthonormal basis for  $L^2(K)$ .
- Let  $L^2(K, \mathbb{C})$  be the set of  $\mu$ -invariant functions  $f : S^1 \times S^1 \longrightarrow \mathbb{C}$  so that

$$\langle f, f \rangle_K = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^\pi f(x, y) \cdot \bar{f}(x, y) dx dy < \infty.$$

## Theorem

Let  $f \in L^2(K, \mathbb{R})$  be a PDF. Then

$$\begin{aligned} & \sum_{m=0}^N \sum_{n=-N}^N \tilde{f}(n, m) \phi_{nm} \\ &= \frac{1}{2\pi^2} + \sum_{m=1}^N a_m (2 \cos my) + \sum_{n=1}^{N_0} b_n (2 \cos 2nx) + c_n (2 \sin 2nx) \\ &+ \sum_{n,m=1}^N d_{nm} \left( 2\sqrt{2} \cos(nx) \cdot \sin \left( my + \frac{\pi}{4} (1 + (-1)^n) \right) \right) \\ &+ \sum_{n,m=1}^N e_{nm} \left( 2\sqrt{2} \sin(nx) \cdot \sin \left( my + \frac{\pi}{4} (1 + (-1)^n) \right) \right) \end{aligned}$$

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converges to  $f$  in the  $L^2$ -norm as  $N \rightarrow \infty$ . Moreover, the functions on the right hand side make up an orthonormal basis  $\mathcal{B}$  for  $L^2(K, \mathbb{R})$ .

## Estimating the Coefficients $v_k(l)$ of $h_k(l)$

Let  $\{P_1, \dots, P_N\} \subseteq K$ , with  $P_r = (X_r, Y_r)$ , be the projection of a random sample of high-contrast  $k \times k$  patches from an image  $I$ .

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For each  $n, m \in \mathbb{N}$  we have unbiased estimators

$$\hat{a}_m = \frac{1}{2N\pi^2} \sum_{r=1}^N \cos(mY_r)$$

$$\hat{b}_n = \frac{1}{2N\pi^2} \sum_{r=1}^N \cos(2nX_r) \quad , \quad \hat{c}_n = \frac{1}{2N\pi^2} \sum_{r=1}^N \sin(2nX_r)$$

$$\hat{d}_{nm} = \frac{1}{\sqrt{2}N\pi^2} \sum_{r=1}^N \cos(nX_r) \cdot \sin\left(mY_r + \frac{\pi}{4}(1 + (-1)^n)\right)$$

$$\hat{e}_{nm} = \frac{1}{\sqrt{2}N\pi^2} \sum_{r=1}^N \sin(nX_r) \cdot \sin\left(mY_r + \frac{\pi}{4}(1 + (-1)^n)\right)$$

converging a.s. to the coefficients w.r.t  $\mathcal{B}$ , of the underlying PDF  $h_k(I) : K \rightarrow \mathbb{R}$ .

# The $v$ -Invariant

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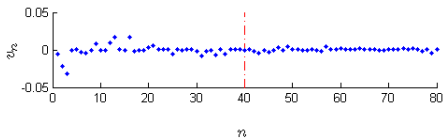
## Definition

The  **$\mathbf{v}$ -Invariant** for  $I$  at scale  $k$  is the vector  $v_k(I)$ .

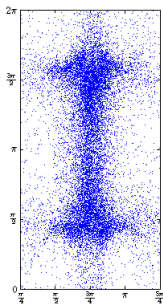
## Examples



$$v_3(\text{Straw}) = (v_1, \dots, v_{40})$$

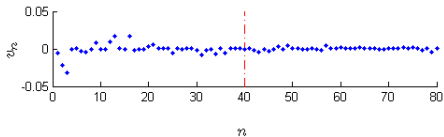


Projected sample

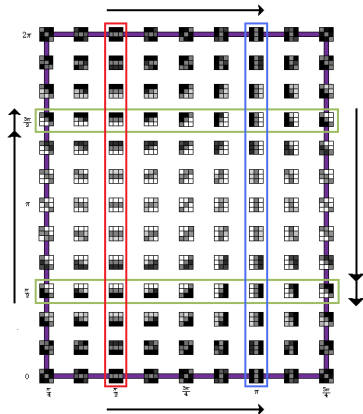
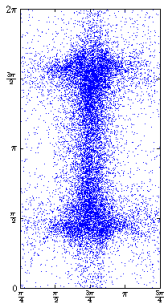




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Projected sample



## A better visualization:

Let  $v_k(I) = (\hat{a}_1, \dots, \hat{a}_m, \hat{b}_n, \hat{c}_n, \hat{d}_{nm}, \hat{e}_{nm}, \dots, \hat{e}_{RS})$  and let  $\hat{h}_k(I) : K \rightarrow \mathbb{R}$  be the function whose coefficients w.r.t.  $\mathcal{B}$  are:

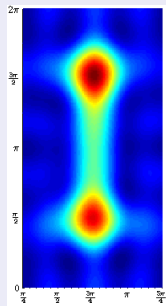
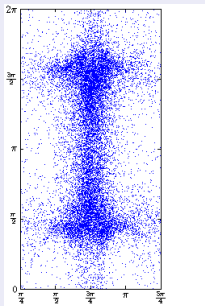
$$\hat{a}_1, \dots, \hat{a}_m, \hat{b}_n, \hat{c}_n, \hat{d}_{nm}, \hat{e}_{nm}, \dots, \hat{e}_{RS}, 0 \dots$$

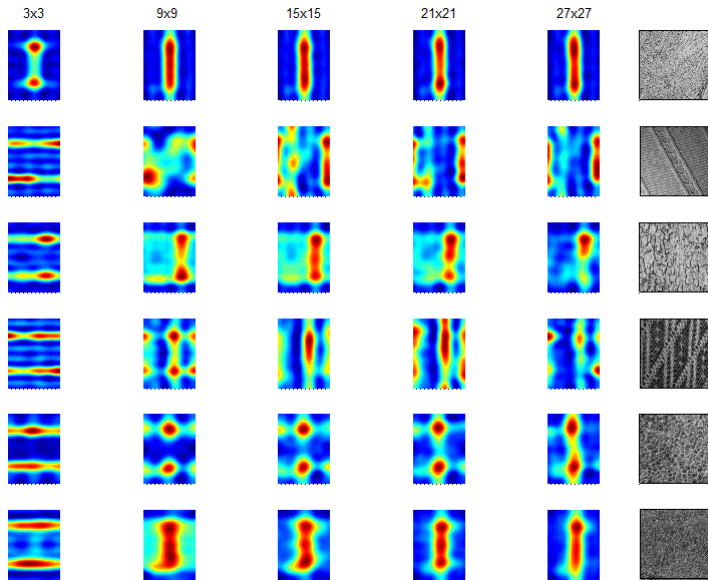
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The Heat-map for  $\hat{h}_k(I)$





# Dissimilarity Measures

$L^2$ -like distance

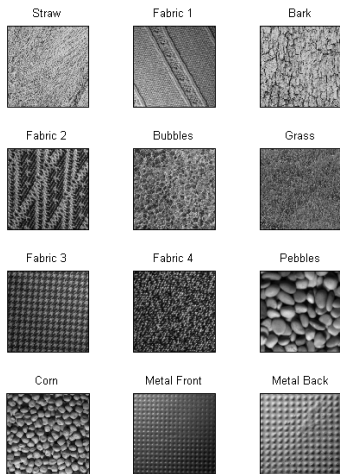
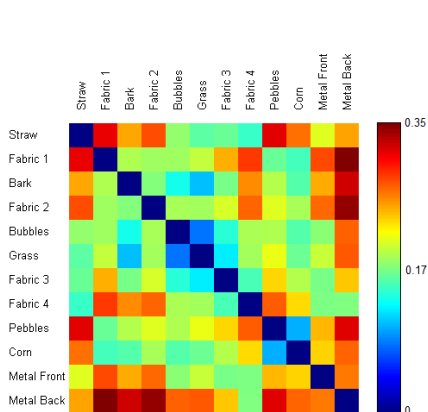
$$\text{Let } d_2(I, J) = \sqrt{\sum_{i=1}^5 \|\hat{h}_{k_i}(I) - \hat{h}_{k_i}(J)\|_{L^2}^2}$$

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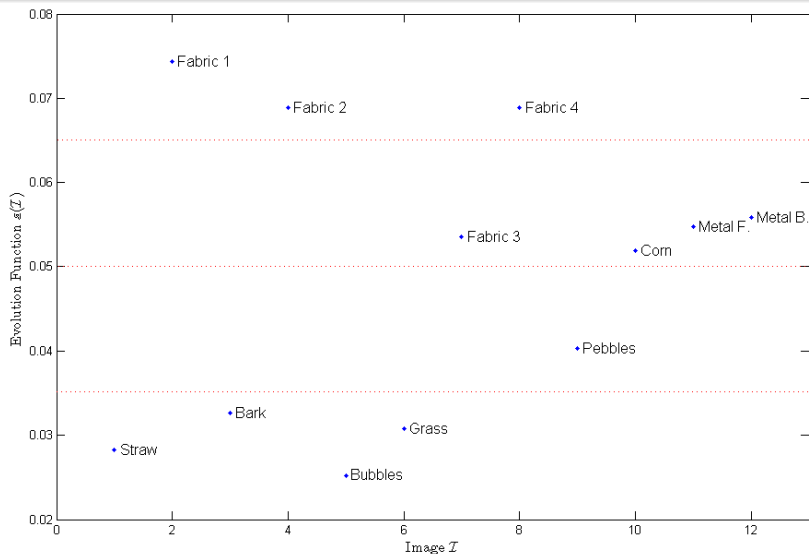
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**The Evolution function:**  $\varepsilon(I) = \sum_{i=1}^4 \|v_{k_i}(I) - v_{k_{i+1}}(I)\|$

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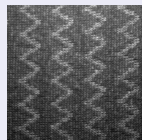


# Image Rotation

Let  $I^\tau$  be the image obtained from  $I$  by a rotation of  $\tau$  degrees.



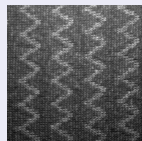
$$I \implies I^{\frac{\pi}{2}}$$



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### Proposition

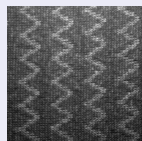
Let  $T \in SL_2(\mathbb{R})$  be rotation by  $\tau \in [-\pi, \pi]$  and let

$$\begin{aligned} v_k(I) &= (a_1, \dots, a_m, b_n, c_n, d_{nm}, e_{nm}, \dots, e_{RS}) \\ v_k(I^\tau) &= (a_1^\tau, \dots, a_m^\tau, b_n^\tau, c_n^\tau, d_{nm}^\tau, e_{nm}^\tau, \dots, e_{RS}^\tau) \end{aligned}$$

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Then  $\widehat{h}_k(I^\tau)(x, y) = \widehat{h}_k(I)(x - \tau, y)$  for all  $(x, y) \in K$  and

$$a_m^\tau = a_m \quad , \quad \begin{bmatrix} c_n^\tau \\ d_n^\tau \end{bmatrix} = T^{2n} \left( \begin{bmatrix} c_n \\ d_n \end{bmatrix} \right) \quad , \quad \begin{bmatrix} d_{nm}^\tau \\ e_{nm}^\tau \end{bmatrix} = T^n \left( \begin{bmatrix} d_{nm} \\ e_{nm} \end{bmatrix} \right).$$

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*The R-distance*

$$d_R(I, J) = \min_{\tau} d_2(I^\tau, J) \quad (1)$$

*defines a (pseudo-)metric.*

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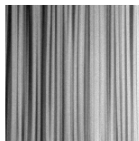
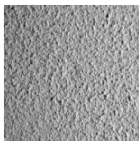
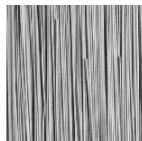
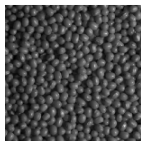
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## Applications...

- Rotation invariance of  $v_k(\cdot)$
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- Classification of images with unknown viewpoint

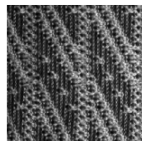
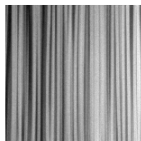
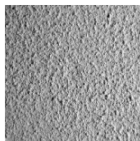
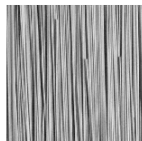
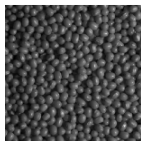
Unknown viewpoint...

Several texture classes



## Unknown viewpoint...

Several texture classes

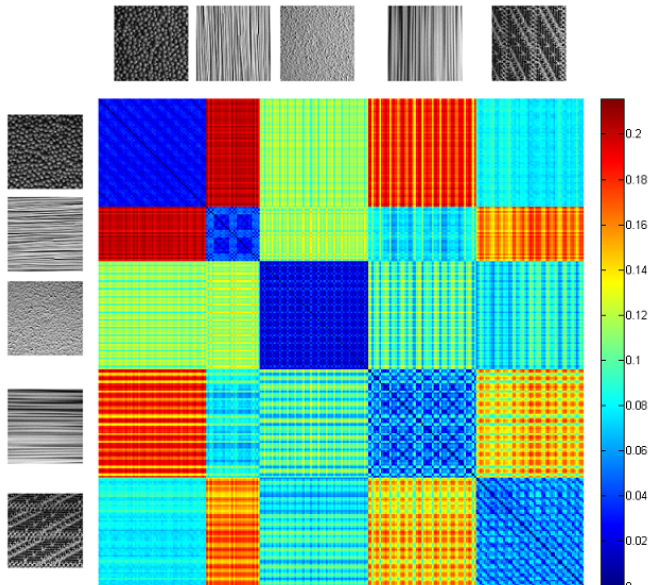


For each texture class: apply surface rotation, unknown light source



a total of 56 (or 28) poses per texture class.

Computing the pairwise distance  $d_R(I, J)$  between the 252 images:



## Two Ideas

- 1 Distributions on the Klein bottle have been around for quite a long time in disguise (responses to filter banks, texton dictionaries), with no mathematical framework, but good results. The  $v$ -Invariant gives both the mathematical framework and a novel, compact representation.
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**Thank you!**